MATH 54 - HINTS TO HOMEWORK 11

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Here are a couple of hints to Homework 11! Enjoy! :)

Warning: This homework is **very** long and very time-consuming! However, it's also very important for the final. So if you have the time, do it thoroughly. But if you have a final project for another class, I would advise you to skip this assignment and perhaps look at it later. Remember the lowest two homeworks are dropped!

Also, for $9.6.9, 9.6.11$, you're allowed to use a calculator. And you're also allowed to use your calculator for 3 other problems of your choice. Other than that, you really have to show your work, especially for finding eigenvalue and eigenvectors. Choose wisely! :)

SECTION 9.4: LINEAR SYSTEMS IN NORMAL FORM

9.4.1, 9.4.3. Those problems are easier to do than to explain. For example, for $9.4.1$:

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}, \mathbf{f} = \begin{bmatrix} t^2 \\ e^t \end{bmatrix}$$

Just beware of the following: If for example $y'(t)$ doesn't contain $x(t)$, then the corresponding term in the matrix A is 0.

9.4.17, 9.4.21. Use the Wronskian! The good news is that the wronskian is very easy to calculate! Just ignore any constants and put all the three vectors in a matrix. For example, for $9.4.17$, the (pre)-Wronskian is:

$$\widetilde{W}(t) = \begin{bmatrix} e^{2t} & e^{2t} & 0 \\ 0 & e^{2t} & e^{3t} \\ 5e^{2t} & -e^{2t} & 0 \end{bmatrix}$$

And as usual, pick your favorite point t_0, and evaluate $det(\widetilde{W}(t_0))$. If this is nonzero, your functions are linearly independent.

9.4.25. Just show $L[\mathbf{x}+\mathbf{y}] = L[\mathbf{x}] + L[\mathbf{y}]$ and $L[c\mathbf{x}] = cL[\mathbf{x}]$, where c is a constant. Use linearity of the derivative and of matrix multiplication.

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9.4.27. Just use the formula $\mathbf{x}(t) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x_0}$ with $t_0 = 0$ and $\mathbf{x}_0 = x(0)$. Notice that you only have to calculate the inverse of $\mathbf{X}$ at 0, **not** in general!

Section 9.5: Homogeneous linear systems with constant coefficients

If you're lost about this, check out the handout 'Systems of differential equations' on my website! Essentially all you have to do is to find the eigenvalues and eigenvectors of A.

Also, to deal with the 'finding the eigenvalues' part, remember the following theorem:

Rational roots theorem: If a polynomial p has a zero of the form $r = \frac{a}{b}$, then a divides the constant term of p and b divides the leading coefficient of p.

This helps you 'guess' a zero of p. Then use long division to factor out p.

9.5.17. First, draw two lines, one spanned by $\mathbf{u_1}$ and the other one spanned by $\mathbf{u_2}$. Then on the first line, draw arrows pointing *away* from the origin (because of the e^{2t}-term in the solution, points on that line *move away* from the origin). On the second line, draw arrows pointing *towards* the origin (because of the e^{-2t}-term, solutions move towards the origin). Finally, for all the other points, all you have to do is to 'connect' the arrows (think of it like drawing a force field or a velocity field).

If you want a picture of how the answer looks like, google 'saddle phase portrait differential equations' and under images, check out the second image you get!

9.5.19, 9.5.21. The fundamental solution set is just the matrix whose columns are the solutions to your differential equation. Basically find the general solution to your differential equation, ignore the constants, and put everything else in a matrix!

9.5.35. For (c), you don't need to derive the relations, just solve the following equation for $\mathbf{u_2}$: $A\mathbf{u_2} = \mathbf{u_1}$.

Section 9.6: Complex eigenvalues

Again, for all those problems, look at the handout 'Systems of differential equations', where everything is discussed in more detail!

9.6.17. First find all the eigenvectors $\mathbf{u_1}, \mathbf{u_2}, \mathbf{u_3}$ of A. Then the general solution is $\mathbf{x}(t) = At^{\lambda_1}\mathbf{u_1} + Bt^{\lambda_2}\mathbf{u_2} + Ct^{\lambda_3}\mathbf{u_3}$, where λ_i is the eigenvalue corresponding to $\mathbf{u_i}$.

9.6.19. Use use equation (10) on page 600 with $m_1 = m_2 = 1$, $k_1 = k_2 = k_3 = 2$. **Careful:** Prof. Grunbaum's method doesn't work here as the k's are $\neq 1$!

Note: The trick where you let $y_1 = x_1$, $y_2 = x_1'$, $y_3 = x_2$, $y_4 = x_2'$ is important to remember! It allows you to convert second-order differential equations into a system of differential equations!

SECTION 9.7: NONHOMOGENEOUS LINEAR EQUATIONS

Again, the handout 'Systems of differential equations' goes through this in more detail!

Note: In what follows, $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ are $2-$vectors.

9.7.3. Guess $\mathbf{f}(t) = e^t \mathbf{a}$

9.7.5. Guess $\mathbf{f}(t) = e^{-2t} \mathbf{a}$

9.7.10. Guess $\mathbf{f}(t) = e^{-t}(\mathbf{a} + t\mathbf{b})$.

9.7.13, 9.7.15. The formula is:

$$\left(\widetilde{W}(t)\right) \begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \mathbf{f}$$

where $\widetilde{W}(t)$ is the (pre)-Wronskian, or fundamental matrix for your system (essentially the solutions but without the constants).

NOTE: The book's answer is **WRONG** (thank you Raphaël for noticing this!)

Either of the following answers is correct:

$$Ae^t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + Be^{-t} \begin{bmatrix} 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 5t - \frac{3}{2}e^{2t} \\ -5t + \frac{9}{2}e^{2t} \end{bmatrix}$$

9.7.27. All that this says is that you have to find a particular solution to $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f_1}$ and $\mathbf{x}'(t) = A\mathbf{x}(t) + \mathbf{f_2}$, where:

$$\mathbf{f_1} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{f_2} = \begin{bmatrix} 0 \\ -e^{-t} \\ -2e^{-t} \end{bmatrix}$$

For the first equation, guess $\mathbf{x}(t) = \mathbf{a}$.

For the second equation, you have to be **very** careful because one of the eigenvalues (-1) coincides with the right-hand-side of your equation. If you look at the previous problem 9.7.26, you see that you have to guess:

$$\mathbf{x}(t) = te^{-t}\mathbf{a} + e^{-t}\mathbf{b}$$

If you plug this into the differential equations and equate the terms with e^{-t} and the terms with te^{-t}, you should get 6 equations in 6 unknowns!

Note: You should get that there are two free variables! (because two of the equations are redundant) In this case, just pick a value for the free variables!

Section 9.8: The matrix exponential functions

Again, check out the 'Systems of differential equations'-handout!

9.8.7. The formula is $e^{At} = \mathbf{X}(t)\mathbf{X}^{-1}(0)$. Remember that you only have to find the inverse at 0, not in general! Or use $e^{At} = Pe^{Dt}P^{-1}$, where D is your matrix of eigenvalues, and P is your matrix of eigenvectors.

Section 10.2: Method of separation of variables

10.2.1, 10.2.3, 10.2.5. Just solve your equation the way you would usually do (for 5, use undetermined coefficients) and plug in the initial conditions. You may or may not find a contradiction! If you find $0 = 0$, that usually means there are infinitely many solutions, depending on your constant A or B.

10.2.9, 10.2.11, 10.2.13. You have to split up your analysis into three cases:

Case 1: $\lambda > 0$. Then let $\lambda = \omega^2$, where $\omega > 0$. This helps you get rid of square roots.

Case 2: $\lambda = 0$.

Case 3: $\lambda < 0$. Then $\lambda = -\omega^2$, where $\omega < 0$.

In each case, solve the equation and plug in your initial condition. You may or may not get a contradiction. Also, remember that y has to be nonzero!

10.2.13. At some point, in the case $\lambda > 0$, you should get:

$$\omega\cos(\omega\pi) + \sin(\omega\pi) = 0$$

At this point, divide your equation by $\cos(\omega\pi)$ to get:

$$\omega + \frac{\sin(\omega\pi)}{\cos(\omega\pi)} = 0$$

That is:

$$\omega + \tan(\omega\pi) = 0$$

If you draw a picture of $\tan(\pi x)$, you might notice that there are infinitely many solutions of $\tan(\pi x) = -x$, hence the above equation has infinitely many solutions ω_m, for each $m = 1, 2, \cdots$. That's how far you have to go to solve the problem!

10.2.21. You **DON'T** have to solve the equation! Just use formula (24) on page 638. Then use formulas (25) and (26) on page 639. And equate coefficients that look alike. For example, the coefficient of $\sin(9x)$ in (26), which corresponds to $n = 9$, should equal to 11.

10.2.27. This is just separation of variables. Plug $u(r, \theta) = R(r)T(\theta)$ into your equation, put all the terms with T on the left-hand-side, and all the terms with R on the right-hand-side, and use the fact that the $LHS = RHS = \lambda$, where λ doesn't depend on r or θ.

10.2.31. This is a strange problem! It's more a logic exercise than a math exercise.

For the first part: $u(r, 0) = 0$ implies $R(r)T(0) = 0$, and provided $R(r) \neq 0$ (which you can do because otherwise $u(r, t) = R(r)T(\theta) = 0$), cancel out $R(r)$ and get $T(0) = 0$. Similar for $T(\pi)$.

For the second part: $u(r, \theta) = R(r)T(\theta)$. Notice that T doesn't depend on r, so what happens as $r \to 0^+$ is solely determined by $R(r)$. So if $R(r)$ is bounded as $r \to 0^+$, then so is u, and if $R(r)$ is unbounded as $r \to 0^+$, then so is u. Since u is bounded as $r \to 0^+$, we get that R has to be bounded as well as $r \to 0^+$.

Section 10.3: Fourier series

10.3.1, 10.3.3, 10.3.5. f is even if $f(-x) = f(x)$, f is odd if $f(-x) = -f(x)$.

10.3.7. Just calculate $fg(-x) = f(-x)g(-x)$

For what follows, use the following formulas:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{T}\right) + b_n \sin\left(\frac{n\pi x}{T}\right) \right\}$$

$$a_n = \frac{1}{T} \int_{-T}^{T} f(x) \cos\left(\frac{n\pi x}{T}\right) dx$$

$$b_n = \frac{1}{T} \int_{-T}^{T} f(x) \sin\left(\frac{n\pi x}{T}\right) dx$$

Where T is such that f is defined on $(-T, T)$

10.3.9. Notice f is odd, so all the a_n are 0.

10.3.11. To calculate the integral, split up the integral from $\int_{-2}^{0} + \int_{0}^{2}$

10.3.17, 10.3.19. The Fourier series converges to $f(x)$ if f is **continuous** at x, and converges to $\frac{f(x^+)+f(x^-)}{2}$ if f is **discontinuous** at x. As for the endpoints T and $-T$, the fourier series converges to the average of f at those endpoints.

10.3.26. Just show:

$$\int_{-1}^{1} \cos\left(\frac{(2m-1)\pi}{2}x\right) \sin\left(\frac{(2n-1)\pi}{2}x\right) dx = 0$$
$$\int_{-1}^{1} \cos\left(\frac{(2m-1)\pi}{2}x\right) \cos\left(\frac{(2n-1)\pi}{2}x\right) dx = 0$$
$$\int_{-1}^{1} \sin\left(\frac{(2m-1)\pi}{2}x\right) \sin\left(\frac{(2n-1)\pi}{2}x\right) dx = 0$$

for all m and n.

Use the following formulas:

$$2\cos(A)\cos(B) = \cos(A+B)+\cos(A-B), 2\sin(A)\sin(B) = \cos(A-B)-\cos(A+B)$$

as well as the fact that odd-ness (for the first one).

10.3.27. Just calculate:

$$\frac{\int_{-1}^{1} f(x)g(x)dx}{\int_{-1}^{1} g(x)^2 dx}$$

for every function $g(x)$ in 10.3.27 (this follows from formula (20) on page 652).

10.3.31. Just show:

$$\int_{-\infty}^{\infty} H_0(x)H_1(x)e^{-x^2} dx = 0$$
$$\int_{-\infty}^{\infty} H_0(x)H_2(x)e^{-x^2} dx = 0$$
$$\int_{-\infty}^{\infty} H_1(x)H_2(x)e^{-x^2} dx = 0$$

For the first one and the third one, use odd-ness. For the second one, split up the integral:

$$4\int_{-\infty}^{\infty} x^2 e^{-x^2} dx - 2\int_{-\infty}^{\infty} e^{-x^2} dx$$

And for the left integral, use integration by parts with $du = xe^{-x^2}$ and $v = x$, which should equal to the right integral.